Categorising set-theoretic multiverses with the multiverse operator

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Ten years ago, Hamkins, 2012 changed the landscape of the foundations of mathematics, by introducing a novel, pluralist conception that tried to clarify some ambiguous notions in current set theoretic practice. In particular, he provided a revolutionary interpretation for the practice of forcing: a *multiverse* of different set theoretic universes. Such an idea immediately sparked an intense debate in the philosophy of set theory and the foundations of mathematics. In the following years, several crucial contributions were made.[1]

Indeed, while the general idea behind pluralism in the philosophy of mathematics is more or less the same every time, the actual mathematical details can vary enormously from one characterisation to the other. Even though all these different set theoretic multiverses share the same, general, philosophical idea, they differ wildly from the mathematical perspective. There are some proposal of assessing all these differences[2], but this research field is still in its infancy.

One of the goals of this research is categorising all the different multiverses, trying to draw distinctions between them and maybe defining some broad categories or types of multiverses. This has been done in several, informal ways in the literature: for example, Antos, S. Friedman, et al., 2015 distinguishes between realist and anti-realist multiverses, appealing to known definition in the ontology of mathematics.

Another possible distinction can be drawn between multiverses in which the truth value of set theoretic statements collapses to the truth value of that same statement in a particular universe of the multiverse *vs* multiverse that don't collapse in this way. Or between *linear* and *branching* multiverses.[3] The linear multiverses expand by building on all the universes part of the multiverse, while the branching ones admits "bifurcations". According to this informal distinction, Steel's set generic multiverse is a linear multiverse, while Hamkins' multiverse is a branching one.

The problem with most of the distinctions found in the literature and folklore on the multiverse is that they are exclusively *philosophically* motivated. While they are very useful in investigating matters in the foundations and philosophy of mathematics, we still lack a purely mathematical characterisation of set theoretic multiverses *as a whole*.

In this paper, I plan to close this gap, and develop a mathematical method to investigate the set-theoretic multiverses as a single, uniform structure. To do so, I introduce the *Multiverse Operator*. With this operator, it's possible to define the structure of all multiverses and then define formal distinction between them. I contend that this changes the landscape of the research in the set theoretic multiverse in an important way. While currently each multiverse is investigated singularly, as an isolated entity, with my proposal it will become possible to approach the class of all multiverses as a single, unified structure. As an analogy, each single multiverse can be thought

[1] See for example Antos, Barton, and S. Friedman, 2021, Martin, 2001, Maddy, 2017, Gitman and Hamkins, 2011, Steel, 2014, and Ternullo and S.-D. Friedman, 2016.

[2] See for example Meadows, 2022.

[3] A recent paper that brings up this distinction, in the context of potentialist systems, is Hamkins and Linnebo, 2022.

of as an algebra, or a logic, while my approach is similar to Universal Algebra, or Universal Logic.[4]

The first step to carry out this program is to characterise the multiverse as single structure. Very briefly, we can characterise each multiverse as a set of models of set theory, in this context called universes. For example, Steel's set generic multiverse is the set of all set-generic extensions of a core universe, Friedman's Hyperuniverse is the set of all countable transitive models of ZFC, etc.. The *Multiverse Operator*, Mlt_i, is a function defined on a stage V_κ of a set theoretic universe.[5] A multiverse operator maps each stage V_κ with the set of all universes, M, that are part of the multiverse generated using V_κ as the ground universe:

$$Mlt_i : V_\kappa \mapsto M.$$

For example, the operator $Mlt_{generic}$ applied to any V_κ, written $Mlt_{generic}(V_\kappa)$, will map to Steel's set generic multiverse. In this way we can define in very general terms what a set theoretic multiverse is: it is an ordered couple (V_κ, Mlt_i), where V_κ is a stage of V and Mlt_i is a multiverse operator.

The multiverse operator can then be used to define the structure of all set theoretic multiverses. To do so, consider the multiverse operator *in general*, without specifying the generating method of each set theoretic multiverse (e.g. the difference between $Mlt_{generic}$ and Mlt_{vlogic}). We can then define the general structure $\langle V_\kappa, Mlt \rangle$, that encompasses all the possible multiverses. This structure forms a *Tarski structure*, that is, it obeys the following axioms:

1. $V_\kappa \subseteq Mlt(V_\kappa)$;
2. $V_\kappa \subseteq V_\lambda \implies Mlt(V_\kappa) \subseteq Mlt(V_\lambda)$;
3. $Mlt(Mlt(V_\kappa)) \subseteq Mlt(V_\kappa)$.

Having defined the structure $\langle V_\kappa, Mlt \rangle$ of set theoretic multiverse, we can start defining the different types of set theoretic multiverses using the multiverse operator.

The first case is the most basic one, i.e. the Single Universe. According the *universism*, the Single Universe is simply V, the cumulative hierarchy. According to *actualism*, it's not possible to extend V in any way: in particular, when we are using set-generic forcing we are not applying it to the whole V, but to a countable transitive model in V. The extension produced by forcing will still be just a countable transitive model inside V. With the multiverse operator, we can say that define the actualist Single Universe as $Mlt_i(V_\kappa) = M[G]$, such that $M[G] \in V_\kappa$ (i.e., no matter the generating method used, we still end up in V). In the same fashion, we can define a height-potentialist (that admits the expansion in *height* of V) Single Universe ($Mlt_i(V_\kappa) = V_{\kappa+1}$), a width potentialist (that admits the expansion in *width* of V) Single Universe ($Mlt_i(V_\kappa) = V_\kappa[G]$), and finally a radical potentialist (that admits the expansion of V both in height and width) Single Universe ($Mlt_i(V_\kappa) = V_{\kappa+1}[G]$). However, this cases aren't actually multiverses cases, so they are not part of the structure $\langle V_\kappa, Mlt \rangle$. This is because they don't satisfy the axiom (3) above.

Back to the case of the proper multiverse, an interesting distinction that we can define is the one between *closed* and *open* multiverses. The closed multiverses are the multiverses that collapse in their ground universe. That is, $Mlt_i(V_\kappa) = V_\kappa$. Multiverses in which the truth value of a statement ϕ in any universe collapses in its truth value in the ground universe are part of this class of multiverses (e.g. Woodin's set generic multiverse is such a multiverse). On the other hand, the open multiverses are the ones that don't collapse in their ground universe, i.e. $Mlt_i(V_\kappa) \neq V_\kappa$.

[4] See for example Beziau, 2007.

[5] Here I am referring to only a stage of the cumulative hierarchy, since we cannot refer to the whole V as an actual mathematical object.

Now consider any universe V_κ such that $Mlt_i(V_\kappa) = V_\kappa$. We can now define the following principle:

$$V_\kappa \models \varphi \iff \text{for any closed multiverse } U_\kappa{}^6 \text{ such that } V_\kappa \subseteq U_\kappa, U_\kappa \models \varphi. \tag{A}$$

This principle says that a universe V_κ, that forms a closed multiverse, witnesses a certain statement φ iff that statement φ is also witnessed by all the other universes that form a closed multiverse and that encompass V_κ. For example, consider Steel's set-generic multiverse. If $V = Ultimate - L$ is true, then the multiverse has a core, and everything that is true in the core is also true in the whole multiverse. Such a multiverse satisfies principle (A): if φ is true in the core of the multiverse, then every extension U_κ of the core also satisfies φ.

Using principle (A), we can then define the linear and branching distinction in the following terms: a multiverse is linear if and only if it principle (A) holds for it, otherwise it's branching. According to this definition, Steel's set generic multiverse is then a linear multiverse.

This is only a first step in a novel research field. It's possible to use the multiverse operator to define all types of distinctions between multiverses. This opens up the possibility to study the set-theoretic multiverse in a more uniform and unified way, instead that trying to assess each single multiverse by itself.

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